A NOTE ON THE ZEROTH PRODUCTS OF FRENKEL-JING OPERATORS

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ABSTRACT. Quantum vertex algebra theory, developed by H.-S. Li, allows us to apply zeroth products of Frenkel-Jing operators, corresponding to Drinfeld realization of $U_q(\widehat{\mathfrak{sl}}_{n+1})$, on the extension of Koyama vertex operators. As a result, we obtain an infinite-dimensional space and describe its structure as a module for the associative algebra $U_q(\mathfrak{sl}_{n+1})_z$, a certain quantum analogue of $U(\mathfrak{sl}_{n+1})$ which we introduce in this paper.

INTRODUCTION

Let $\mathfrak{g}$ be a simple Lie algebra with the root lattice Q and weight lattice P. Denote by $\widehat{\mathfrak{g}}$ the associated (untwisted) affine Lie algebra on the underlying vector space

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

with the bracket relations defined in a usual way (for details see [9]). The induced $\widehat{\mathfrak{g}}$-module

$$V = V_{\widehat{\mathfrak{g}}}(l, 0) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_{(\leq 0)})} \mathbb{C}_l, \quad l \in \mathbb{N},$$

has a vertex operator algebra structure

$$\begin{array}{rcl} V & \to & \operatorname{Hom}(V, V((z))) \\ v & \mapsto & Y(v, z) = \displaystyle\sum_{r \in \mathbb{Z}} v_r z^{-r-1} \end{array}$$

(for details and notation see [12]). Denote by $a_r = a(r)$ the action of $a \otimes t^r$, $a \in \mathfrak{g}$, $r \in \mathbb{Z}$, on an arbitrary restricted $\widehat{\mathfrak{g}}$-module W and set

$$a_W(z) = \sum_{r \in \mathbb{Z}} a_r z^{-r-1} \in \operatorname{Hom}(W, W((z))).$$

Every restricted $\widehat{\mathfrak{g}}$-module W of level l is a module for vertex algebra V, and vice versa, which gives us correspondence

$$a^{(1)}(r_1) \cdots a^{(m)}(r_m)1 \qquad \longleftrightarrow \qquad a_W^{(1)}(z)_{r_1} \cdots a_W^{(m)}(z)_{r_m} 1_W, \tag{0.1}$$

$a^{(j)} \in \mathfrak{g}$, $r_j \in \mathbb{Z}$, $j \geq 0$, between $\widehat{\mathfrak{g}}$-module actions and products of the local vertex operators. The quotient of $V_{\widehat{\mathfrak{g}}}(l, 0)$ over its (unique) largest proper ideal $I_{\widehat{\mathfrak{g}}}(l, 0)$,

$$L_{\widehat{\mathfrak{g}}}(l, 0) = V_{\widehat{\mathfrak{g}}}(l, 0) / I_{\widehat{\mathfrak{g}}}(l, 0),$$

is a simple vertex operator algebra whose inequivalent irreducible modules $L_{\widehat{\mathfrak{g}}}(l, L(\lambda))$, where $L(\lambda)$ is the irreducible $\mathfrak{g}$-module with the integral dominant highest weight λ, are exactly level l irreducible highest weight integrable $\widehat{\mathfrak{g}}$-modules.

Consider the space

$$V_L = M(1) \otimes \mathbb{C}\{L\},$$

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where $M(1)$ is the irreducible level 1 Heisenberg algebra module and $\mathbb{C}\{L\}$ is the group algebra of the lattice $L = Q, P$. In the construction of vertex operator algebras and modules associated to even lattices, the space V_Q is equipped with a vertex algebra structure, while V_P becomes its module. The simple vertex algebra V_Q is isomorphic to $L_{\widehat{\mathfrak{g}}}(1,0)$ while V_P is a direct sum of the irreducible V_Q-modules. This construction provides us with the realizations of

$$x_\alpha(z) = \sum_{r\in\mathbb{Z}} x_\alpha(r) z^{-r-1} = Y(e^\alpha, z)$$

for Chevalley generators $x_\alpha \in \mathfrak{g}$, $\alpha \in Q$, as well as as with the realizations of the intertwining operators.

The main motivation for this research is the fact that the top of every irreducible highest weight integrable $\widehat{\mathfrak{g}}$-module $L_{\widehat{\mathfrak{g}}}(l, L(\lambda))$ is the irreducible $\mathfrak{g}$-module $L(\lambda)$, where the action of $a \in \mathfrak{g} \subset L_{\widehat{\mathfrak{g}}}(l, 0)$ is given as a_0.

We consider quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_{n+1})$, defined in terms of Drinfeld generators, coefficients of formal Laurent series $x_j^{\pm}(z)$, $\psi_j(z)$, $\phi_j(z)$, $j = 1, 2, \ldots, n$ (see [3]). Even though there is no correspondence as in (0.1) for $U_q(\widehat{\mathfrak{sl}}_{n+1})$, some of the main ingredients of the abovementioned constructions do exist. For example, Frenkel-Jing realization of level 1 integrable highest weight $U_q(\widehat{\mathfrak{sl}}_{n+1})$-modules (see [6]) gives us the quantum analogues of the operators $x_\alpha(z) = Y(e^\alpha, z)$, while Y. Koyama in [10] found the realization of intertwining operator $\mathcal{Y}_i(z) = \mathcal{Y}(e^{\lambda_i}, z)$, $i = 1, 2, \ldots, n$. All of these operators are defined on the space

$$V = M(1) \otimes \mathbb{C}\{P\},$$

the tensor product of level 1 irreducible q-Heisenberg algebra module $M(1)$ and the twisted group algebra $\mathbb{C}\{P\}$ of the (classical) weight lattice P.

Since the operators $x_j^{\pm}(z)$, $\psi_j(z)$, $\phi_j(z)$ are not local, we can not multiply them as in the classical case, using the theory of local vertex operators. However, they satisfy the notion of quasi compatibility, i.e. for any two such operators $a(z), b(z) \in \operatorname{Hom}(V, V((z)))$ there exists a nonzero polynomial $p(z_1, z_2)$ such that

$$p(z_1, z_2) a(z_1) b(z_2) \in \operatorname{Hom}(V, V((z_1, z_2))).$$

H.-S. Li introduced rth products, $r \in \mathbb{Z}$, among quasi compatible operators,

$$Y_{\mathcal{E}}(a(z), z_0) b(z) = \sum_{r\in\mathbb{Z}} (a(z)_r b(z)) z_0^{-r-1} \in (\operatorname{End} V)[[z_0^{\pm1}, z^{\pm1}]],$$

and proved that a family of such operators (acting on an arbitrary level $t \in \mathbb{C}$ restricted module) generates a (weak) quantum vertex algebra (cf. [13], [14], [15]). We would like to mention that there exist several other approaches to development of "quantum vertex algebra theories". For more information the reader may consult [1], [2], [5], [7].

We consider the set

$$\left\{x_j^{\pm}(zq^t), \psi_j(zq^t), \phi_j(zq^t) : j = 1, 2, \ldots, n,\ t \in \tfrac{1}{2}\mathbb{Z}\right\} \subset \operatorname{Hom}(V, V((z))).$$

Our goal is to study zeroth products of these operators acting on $\mathcal{Y}_i(z)$. Since zeroth products $a(zq^t)_0 b(z)$ equal to zero for almost all t, we introduce an action "$\bullet$", which plays a role of zeroth products from the (classical) vertex algebra theory in this quantum setting. We show that the "$\bullet$" products similar to the right side of (0.1) are equal to zero if and only if the (zeroth) products similar to the left side of (0.1), i.e. the products of Chevalley generators, are equal to zero.

For a quasi compatible pair $(a(z), b(z))$ we define

$$a(z)_\bullet b(z) = \sum_{t\in\frac{1}{2}\mathbb{Z}} \left(\frac{1}{zq^t}\right)^{(\mathrm{wt}\, a(z), \mathrm{wt}\, b(z))+1} q^t a(zq^t)_0 b(z). \tag{0.2}$$

Our main point of interest is a $\langle\mathcal{Y}_i(z)\rangle$, a vector space over $\mathbb{C}(q^{1/2})$ spanned by all the vectors obtained by the above defined action "$\bullet$" of the operators $x_j^\pm(z)$, $\psi_j(z)$, $\phi_j(z)$, $j = 1, 2, \dots, n$, on $\mathcal{Y}_i(z)$. Having in mind the classical case, one may expect $\langle\mathcal{Y}_i(z)\rangle$ to be, roughly speaking, a quantum version of the irreducible $\mathfrak{sl}_{n+1}$-modules $L(\lambda_i)$.

In the first section we recall some preliminary results, while in the second section we study the action (0.2) of Frenkel-Jing operators. Our key result here is the following Lemma, which plays an important role in the proof of our main result, given in the third section:

Lemma 2.14 *For any $a(z) \in \langle\mathcal{Y}_i(z)\rangle$ we have*

$$x_{j_1}^+(z)_\bullet x_{j_2}^-(z)_\bullet a(z) - x_{j_2}^-(z)_\bullet x_{j_1}^+(z)_\bullet a(z) = \frac{\delta_{j_1 j_2}}{q - q^{-1}} \left(\psi_{j_1}(z) - \phi_{j_1}(z)\right)_\bullet a(z).$$

We also identify a basis $\mathcal{B}_i$ for the space $\langle\mathcal{Y}_i(z)\rangle$ as a set, which consists of some of the operators

$$x_{l_1}^-(z)_\bullet \cdots x_{l_r}^-(z)_\bullet \mathcal{Y}_i(z)\psi_{j_1}(zq^{t_1}) \cdots \psi_{j_s}(zq^{t_s}).$$

In the last, third section we introduce a certain $\mathbb{C}(q^{1/2})$ subalgebra $U_q(\mathfrak{sl}_{n+1})_z$ of the algebra $U_q(\mathfrak{sl}_{n+1})[[z_1, \dots, z_n]]$, given in terms of generators $\bar{e}_j$, f_j, $\bar{k}_j$, $j = 1, 2, \dots, n$. Roughly speaking, this new algebra may be considered as a quantum analogue of $U(\mathfrak{sl}_{n+1})$, since its classical limit is equal to $U(\mathfrak{sl}_{n+1})$. Next, we construct some infinite-dimensional $U_q(\mathfrak{sl}_{n+1})_z$-modules $L(\lambda_i)_z$ corresponding to the (finite-dimensional) irreducible $U_q(\mathfrak{sl}_{n+1})$-modules $L(\lambda_i)$ with the integral dominant highest weight λ_i. The main result of this paper is Theorem 3.3, which identifies $\langle\mathcal{Y}_i(z)\rangle$ as a module for $U_q(\mathfrak{sl}_{n+1})_z$ and establishes an $U_q(\mathfrak{sl}_{n+1})_z$-module isomorphism $L(\lambda_i)_z \cong \langle\mathcal{Y}_i(z)\rangle$:

Theorem 3.3 *(1) There exists a structure of $U_q(\mathfrak{sl}_{n+1})_z$-module on the space $\langle\mathcal{Y}_i(z)\rangle$ such that*

$$\bar{e}_j a(z) = x_j^+(z)_\bullet a(z),$$
$$f_j a(z) = x_j^-(z)_\bullet a(z),$$
$$\bar{k}_j a(z) = \psi_j(z)_\bullet a(z)$$

for all $j = 1, 2, \dots, n$ and $a(z) \in \langle\mathcal{Y}_i(z)\rangle$.
(2) $U_q(\mathfrak{sl}_{n+1})_z$-modules $L(\lambda_i)_z$ and $\langle\mathcal{Y}_i(z)\rangle$ are isomorphic.

1. Preliminaries

1.1. **Quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_{n+1})$.** We recall some facts from the theory of affine Kac-Moody Lie algebras (see [9] for more details). Let $\widehat{A} = (a_{ij})_{i,j=0}^n$ be a generalized Cartan matrix of (affine) type $A_n^{(1)}$ associated with the affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}}_{n+1}$. Let $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{sl}}_{n+1}$ be a vector space over $\mathbb{C}$ with a basis consisting of simple coroots $\alpha_j^\vee$, $j = 0, 1, \dots, n$, and derivation d. Denote by $\alpha_0, \alpha_1, \dots, \alpha_n$ simple roots, i.e. linear functionals from $\widehat{\mathfrak{h}}^*$ such that

$$\alpha_i(\alpha_j^\vee) = a_{ji}, \quad \alpha_i(d) = \delta_{i0}, \quad i, j = 0, 1, \dots, n.$$

The invariant symmetric bilinear form on $\widehat{\mathfrak{h}}^*$ is given by

$$(\alpha_i, \alpha_j) = a_{ij}, \quad (\delta, \alpha_i) = (\delta, \delta) = 0, \quad i, j = 0, 1, \dots, n.$$

Denote by $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ fundamental weights, elements of $\widehat{\mathfrak{h}}^*$ such that

$$\Lambda_i(\alpha_j^\vee) = \delta_{ij}, \quad \Lambda_i(d) = 0, \quad i, j = 0, 1, \ldots, n.$$

The center of the Lie algebra $\widehat{\mathfrak{sl}}_{n+1}$ is one-dimensional and it is generated by the element

$$c = \alpha_0^\vee + \alpha_1^\vee + \ldots + \alpha_n^\vee \in \widehat{\mathfrak{h}}$$

and imaginary roots of $\widehat{\mathfrak{sl}}_{n+1}$ are integer multiples of

$$\delta = \alpha_0 + \alpha_1 + \ldots + \alpha_n \in \widehat{\mathfrak{h}}^*.$$

Define a weight lattice $\widehat{P}$ as a free Abelian group generated by the elements $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ and δ. An integral dominant weight is any $\Lambda \in \widehat{P}$ such that $\Lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0}$ for $i = 0, 1, \ldots, n$.

Let $\mathfrak{h} \subset \widehat{\mathfrak{h}}$ be Cartan subalgebra of the simple Lie algebra $\mathfrak{sl}_{n+1}$, generated by the elements $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee$. Denote by

$$Q = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i \subset \mathfrak{h}^* \quad \text{and} \quad P = \bigoplus_{i=1}^{n} \mathbb{Z}\lambda_i \subset \mathfrak{h}^*$$

the classical root lattice and the classical weight lattice, where elements λ_i satisfy

$$\lambda_i(\alpha_j^\vee) = \delta_{ij}, \quad i, j = 1, 2, \ldots, n.$$

Fix an indeterminate q. For any two integers m and k, $k > 0$, define q-integers,

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}},$$

and q-factorials,

$$[0]! = 1, \quad [k]! = [k][k-1]\cdots[1].$$

For all nonnegative integers m and k, $m \geq k$, define q-binomial coefficients,

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{[m]!}{[k]![m-k]!}.$$

Definition 1.1 The quantized enveloping algebra $U_q(\mathfrak{sl}_{n+1})$ is the associative algebra over $\mathbb{C}(q^{1/2})$ with unit 1 generated by the elements e_i, f_i and $K_i^{\pm 1}$, $i = 1, 2, \ldots, n$, subject to the following relations:

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{Q1}$$

$$K_i e_j K_i^{-1} = q^{(\alpha_i, \alpha_j)} e_j, \quad K_i f_j K_i^{-1} = q^{-(\alpha_i, \alpha_j)} f_j, \tag{Q2}$$

$$[e_i, f_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{Q3}$$

$$\sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix} e_i^{m-s} e_j e_i^s = 0, \quad \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix} f_i^{m-s} f_j f_i^s = 0 \quad \text{for } i \neq j, \tag{Q4}$$

where $m = 1 - a_{ij}$.

Next, we present Drinfeld realization of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_{n+1})$.

Definition 1.2 ([3]) The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_{n+1})$ is the associative algebra over $\mathbb{C}(q^{1/2})$ with unit 1 generated by the elements $x_i^{\pm}(r)$, $a_i(s)$, $K_i^{\pm 1}$, $\gamma^{\pm 1/2}$ and $q^{\pm d}$,

$i = 1, 2, \ldots, n$, $r, s \in \mathbb{Z}$, $s \neq 0$, subject to the following relations:

$$[\gamma^{\pm 1/2}, u] = 0 \text{ for all } u \in U_q(\widehat{\mathfrak{g}}), \tag{D1}$$

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{D2}$$

$$[a_i(k), a_j(l)] = \delta_{k+l\,0} \frac{[a_{ij}k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \tag{D3}$$

$$[a_i(k), K_j^{\pm 1}] = [q^{\pm d}, K_j^{\pm 1}] = 0, \tag{D4}$$

$$q^d x_i^{\pm}(k) q^{-d} = q^k x_i^{\pm}(k), \quad q^d a_i(k) q^{-d} = q^k a_i(k), \tag{D5}$$

$$K_i x_j^{\pm}(k) K_i^{-1} = q^{\pm(\alpha_i, \alpha_j)} x_j^{\pm}(k), \tag{D6}$$

$$[a_i(k), x_j^{\pm}(l)] = \pm \frac{[a_{ij}k]}{k} \gamma^{\mp |k|/2} x_j^{\pm}(k+l), \tag{D7}$$

$$\begin{aligned} &x_i^{\pm}(k+1) x_j^{\pm}(l) - q^{\pm(\alpha_i, \alpha_j)} x_j^{\pm}(l) x_i^{\pm}(k+1) \\ &\qquad = q^{\pm(\alpha_i, \alpha_j)} x_i^{\pm}(k) x_j^{\pm}(l+1) - x_j^{\pm}(l+1) x_i^{\pm}(k), \end{aligned} \tag{D8}$$

$$[x_i^{+}(k), x_j^{-}(l)] = \frac{\delta_{ij}}{q - q^{-1}} \left(\gamma^{\frac{k-l}{2}} \psi_i(k+l) - \gamma^{\frac{l-k}{2}} \phi_i(k+l) \right), \tag{D9}$$

$$\operatorname*{Sym}_{l_1, l_2, \ldots, l_m} \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix} x_i^{\pm}(l_1) \cdots x_i^{\pm}(l_s) x_j^{\pm}(k) x_i^{\pm}(l_{s+1}) \cdots x_i^{\pm}(l_m) = 0 \quad \text{for } i \neq j, \tag{D10}$$

where $m = 1 - a_{ij}$. The elements $\phi_i(-r)$ and $\psi_i(r)$, $r \in \mathbb{Z}_{\geq 0}$, are given by

$$\phi_i(z) = \sum_{r=0}^{\infty} \phi_i(-r) z^r = K_i^{-1} \exp\left(-(q - q^{-1}) \sum_{r=1}^{\infty} a_i(-r) z^r \right),$$

$$\psi_i(z) = \sum_{r=0}^{\infty} \psi_i(r) z^{-r} = K_i \exp\left((q - q^{-1}) \sum_{r=1}^{\infty} a_i(r) z^{-r} \right).$$

Denote by $x_i^{\pm}(z)$ the series

$$x_i^{\pm}(z) = \sum_{r \in \mathbb{Z}} x_i^{\pm}(r) z^{-r-1} \in U_q(\widehat{\mathfrak{sl}}_{\mathfrak{n}+1})[[z^{\pm 1}]]. \tag{1.1}$$

We shall continue to use the notation $x_i^{\pm}(z)$ for the action of the expression (1.1) on an arbitrary $U_q(\widehat{\mathfrak{sl}}_{n+1})$-module V:

$$x_i^{\pm}(z) = \sum_{r \in \mathbb{Z}} x_i^{\pm}(r) z^{-r-1} \in (\operatorname{End} V)[[z^{\pm 1}]].$$

Some basic facts about $U_q(\widehat{\mathfrak{sl}}_{n+1})$ (and its representation theory) can be found in [8].

1.2. **Frenkel-Jing realization.** We present Frenkel-Jing realization of the integrable highest weight $U_q(\widehat{\mathfrak{sl}}_{n+1})$-modules $L(\Lambda_i)$, $i = 0, 1, \ldots, n$ (see [6]).

Let V be an arbitrary $U_q(\widehat{\mathfrak{sl}}_{n+1})$-module of level c. The Heisenberg algebra $U_q(\widehat{\mathfrak{h}})$ of level c is generated by the elements $a_i(k)$, $i = 1, 2, \ldots, n$, $k \in \mathbb{Z} \setminus \{0\}$ and the central element $\gamma^{\pm 1} = q^{\pm c}$ subject to the relations

$$[a_i(r), a_j(s)] = \delta_{r+s\,0} \frac{[a_{ij} r][cr]}{r}, \quad i, j = 1, 2, \ldots, n,\ r, s \in \mathbb{Z} \setminus \{0\}. \tag{1.2}$$

The Heisenberg algebra $U_q(\widehat{\mathfrak{h}})$ has a natural realization on the space $\operatorname{Sym}(\widehat{\mathfrak{h}}^-)$ of the symmetric algebra generated by the elements $a_i(-r)$, $r \in \mathbb{Z}_{>0}$, $i = 1, 2, \ldots, n$, via the

following rule:

$$\begin{aligned}\gamma^{\pm 1} &\ \dots\ \text{multiplication by } q^{\pm c},\\ a_i(r) &\ \dots\ \text{differentiation operator subject to (1.2)},\\ a_i(-r) &\ \dots\ \text{multiplication by the element } a_i(-r).\end{aligned}$$

Denote the resulted irreducible $U_q(\widehat{\mathfrak{h}})$-module as $K(c)$. Define the following operators on $K(c)$:

$$E_-^{\pm}(a_i, z) = \exp\left(\mp \sum_{r\geq 1} \frac{q^{\mp cr/2}}{[cr]} a_i(-r) z^r\right),$$

$$E_+^{\pm}(a_i, z) = \exp\left(\pm \sum_{r\geq 1} \frac{q^{\mp cr/2}}{[cr]} a_i(r) z^{-r}\right).$$

The associative algebra $\mathbb{C}\{P\}$ is generated by the elements $e^{\alpha_2}, e^{\alpha_3}, \dots e^{\alpha_n}$ and e^{λ_n} subject to the relations

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i,\alpha_j)} e^{\alpha_j} e^{\alpha_i}, \quad e^{\alpha_i} e^{\lambda_n} = (-1)^{\delta_{in}} e^{\lambda_n} e^{\alpha_i}, \qquad i, j = 2, 3, \dots, n.$$

For $\alpha = m_2\alpha_2 + \dots + m_n\alpha_n + m_{n+1}\lambda_n \in P$ denote $e^{m_2\alpha_2} \cdots e^{m_n\alpha_n} e^{m_{n+1}\lambda_n} \in \mathbb{C}\{P\}$ by e^{α}. Denote by $\mathbb{C}\{Q\}$ the subalgebra of $\mathbb{C}\{P\}$ generated by the elements e^{α_i}, $i = 1, 2, \dots, n$. Set

$$\mathbb{C}\{Q\} e^{\lambda_i} = \left\{ a e^{\lambda_i} \,|\, a \in \mathbb{C}\{Q\} \right\}.$$

For $\alpha \in Q$ define an action z^{∂_α} on $\mathbb{C}\{Q\} e^{\lambda_i}$ by

$$z^{\partial_\alpha} e^{\beta} e^{\lambda_i} = z^{(\alpha, \beta+\lambda_i)} e^{\beta} e^{\lambda_i}.$$

Theorem 1.3 ([6]) *By the action*

$$x_j^{\pm}(z) := E_-^{\pm}(-a_j, z) E_+^{\pm}(-a_j, z) \otimes e^{\pm\alpha_j} z^{\pm\partial_{\alpha_j}},$$

$j = 1, 2, \dots, n$, the space

$$K(1) \otimes \mathbb{C}\{Q\} e^{\lambda_i}$$

becomes the integrable highest weight module of $U_q(\widehat{\mathfrak{sl}}_{n+1})$ with the highest weight Λ_i.

1.3. **Operator $\mathcal{Y}_i(z)$.** In [10] Koyama found a realization of vertex operators for level one integrable highest weight modules of $U_q(\widehat{\mathfrak{sl}}_{n+1})$. In [11] we used similar operators $\mathcal{Y}_i(z) = \mathcal{Y}(e^{\lambda_i}, z)$, which we now briefly recall. Let

$$V = K(1) \otimes \mathbb{C}\{P\}.$$

For $i = 1, 2, \dots, n$ and $l \in \mathbb{Z}$, $l \neq 0$, we define elements $a_i^*(l) \in U_q(\widehat{\mathfrak{h}})$ by

$$a_i^*(l) = m_i^{(1)} a_1(l) + m_i^{(2)} a_2(l) + \dots + m_i^{(n)} a_n(l),$$

where

$$m_i^{(j)} = \begin{cases} \dfrac{[jl][(n-i+1)l]}{[(n+1)l][l]} & \text{for } j \leq i; \\[2ex] \dfrac{[il][(n-j+1)l]}{[(n+1)l][l]} & \text{for } j > i. \end{cases}$$

For $i, j = 1, 2, \dots, n$, $l, k \in \mathbb{Z}$, $l, k \neq 0$ we have

$$[a_i^*(l), a_j(k)] = \delta_{ij} \delta_{l+k\,0} \frac{[l]^2}{l}.$$

Define the following operators on the space V:

$$E_-(a_i^*, z) = \exp\left(\sum_{r=1}^{\infty} \frac{q^{r/2}}{[r]} a_i^*(-r) z^r\right),$$

$$E_+(a_i^*, z) = \exp\left(-\sum_{r=1}^{\infty} \frac{q^{r/2}}{[r]} a_i^*(r) z^{-r}\right).$$

Definition 1.4 We define an operator $\mathcal{Y}_i(z) \in \mathrm{Hom}(V, V((z^{1/(n+1)})))$ by

$$\mathcal{Y}_i(z) = E_-(a_i^*, z) E_+(a_i^*, z) \otimes e^{\lambda_i} (-1)^{(1-\delta_{in}) i \partial_{\lambda_n}} z^{\partial_{\lambda_i}}.$$

By applying the operator $\mathcal{Y}_i(z)$ on an arbitrary vector $v \in V$, we get a formal power series in fractional powers $z^{\frac{1}{n+1}}$ of the variable z, that has only a finite number of negative powers.

The relations in the next proposition can be proved by a direct calculation. Some of them can be found in [4] and [11].

Proposition 1.5 *For any $i, j = 1, 2, \ldots, n$ the following relations hold on $V = K(1) \otimes \mathbb{C}\{P\}$:*

$$x_i^-(z_1) x_i^-(z_2) = (z_1 - z_2)(z_1 - q^2 z_2) : x_i^-(z_1) x_i^-(z_2) : \tag{1.3}$$

$$(z_1 - q z_2) x_i^-(z_1) x_j^-(z_2) =: x_i^-(z_1) x_j^-(z_2) : \quad \text{for } |i-j| = 1 \tag{1.4}$$

$$x_i^-(z_1) x_j^-(z_2) =: x_i^-(z_1) x_j^-(z_2) : \quad \text{for } |i-j| > 1 \tag{1.5}$$

$$(z_1 - q z_2) x_i^-(z_1) \mathcal{Y}_i(z_2) =: x_i^-(z_1) \mathcal{Y}_i(z_2) : \tag{1.6}$$

$$x_i^-(z_1) \mathcal{Y}_j(z_2) =: x_i^-(z_1) \mathcal{Y}_j(z_2) : \quad \text{for } i \neq j \tag{1.7}$$

$$(z_1 - q^{-3/2} z_2) \psi_i(z_1) x_i^-(z_2) = (q^{-2} z_1 - q^{1/2} z_2) x_i^-(z_1) \psi_i(z_2) \tag{1.8}$$

$$(z_1 - q^{3/2} z_2) \psi_i(z_1) x_j^-(z_2) = (q z_1 - q^{1/2} z_2) x_j^-(z_1) \psi_i(z_2) \quad \text{for } |i-j| = 1 \tag{1.9}$$

$$\psi_i(z_1) x_j^-(z_2) = x_j^-(z_1) \psi_i(z_2) \quad \text{for } |i-j| > 1 \tag{1.10}$$

$$(z_1 - q^{3/2} z_2) \psi_i(z_1) \mathcal{Y}_i(z_2) = (q z_1 - q^{1/2} z_2) \mathcal{Y}_i(z_2) \psi_i(z_1) \tag{1.11}$$

$$\psi_i(z_1) \mathcal{Y}_j(z_2) = \mathcal{Y}_j(z_2) \psi_i(z_1) \quad \text{for } i \neq j \tag{1.12}$$

$$(z_1 - q z_2)(z_1 - q^{-1} z_2) x_i^+(z_1) x_i^-(z_2) =: x_i^+(z_1) x_i^-(z_2) : \tag{1.13}$$

$$x_i^+(z_1) x_j^-(z_2) = (z_1 - z_2) : x_i^+(z_1) x_j^-(z_2) : \quad \text{for } |i-j| = 1 \tag{1.14}$$

$$x_i^+(z_1) x_j^-(z_2) =: x_i^+(z_1) x_j^-(z_2) : \quad \text{for } |i-j| > 1 \tag{1.15}$$

$$x_i^+(z_1) \mathcal{Y}_i(z_2) = (z_1 - z_2) : x_i^+(z_1) \mathcal{Y}_i(z_2) : \tag{1.16}$$

$$x_i^+(z_1) \mathcal{Y}_j(z_2) =: x_i^+(z_1) \mathcal{Y}_j(z_2) : \quad \text{for } i \neq j. \tag{1.17}$$

2. Space $\langle \mathcal{Y}_i(z) \rangle$

2.1. **The action of Frenkel-Jing operators.** Let V be an arbitrary vector space over $\mathbb{C}(q^{1/2})$. We recall two definitions from [13].

Definition 2.1 An ordered pair $(a(z), b(z))$ in $\mathrm{Hom}(V, V((z)))$ is said to be quasi compatible if there exist a nonzero polynomial $p(z_1, z_2) \in \mathbb{C}(q^{1/2})[z_1, z_2]$ such that

$$p(z_1, z_2) a(z_1) b(z_2) \in \mathrm{Hom}(V, V((z_1, z_2))).$$

Denote by $\mathbb{C}(q^{1/2})_*(z, z_0)$ the extension of the algebra $\mathbb{C}(q^{1/2})[[z, z_0]]$ by inverses of nonzero polynomials. Let ι_{z,z_0} be a unique algebra embedding

$$\iota_{z,z_0} \colon \mathbb{C}(q^{1/2})_*(z, z_0) \to \mathbb{C}(q^{1/2})((z))((z_0))$$

that extends the identity endomorphism of $\mathbb{C}(q^{1/2})[[z, z_0]]$.

Definition 2.2 Let $(a(z), b(z))$ be a quasi compatible (ordered) pair in $\mathrm{Hom}(V, V((z)))$. For $r \in \mathbb{Z}$ we define $a(z)_r b(z) \in (\mathrm{End}\, V)[[z^{\pm 1}]]$ in terms of generating function

$$Y_{\mathcal{E}}(a(z), z_0)b(z) = \sum_{r\in\mathbb{Z}} (a(z)_r b(z)) z_0^{-r-1} \in (\mathrm{End}\, V)[[z_0^{\pm 1}, z^{\pm 1}]]$$

by

$$Y_{\mathcal{E}}(a(z), z_0)b(z) = \iota_{z,z_0}\left(p(z_0 + z, z)^{-1}\right)\left(p(z_1, z)a(z_1)b(z)\right)\Big|_{z_1=z+z_0}, \tag{2.1}$$

where $p(z_1, z_2) \in \mathbb{C}(q^{1/2})[z_1, z_2]$ is any nonzero polynomial such that

$$p(z_1, z_2)a(z_1)b(z_2) \in \mathrm{Hom}(V, V((z_1, z_2))).$$

The Proposition 1.5 implies that the each ordered pair consisting of the operators $x_j^{\pm}(z)$, $\psi_j(z)$, $\phi_j(z)$ is quasi compatible, so the rth product, $r \in \mathbb{Z}$, (given by Definition 2.2) is well defined for such a pair.

Remark 2.3 The expressions (1.6), (1.7), (1.11), (1.12), (1.16), (1.17) in Proposition 1.5 are not elements of $\mathrm{Hom}(V, V((z_1, z_2)))$, because they consist of rational powers of z_2. However, they are obviously contained in $\mathrm{Hom}(V, V((z_1, z_2^{1/(n+1)})))$, so Definition 2.2 may be applied. More precisely, the rth products $a(z)_r\mathcal{Y}_i(z)$ for $a(z) = x_j^{\pm}(z), \psi_j(z), \phi_j(z)$ are well defined elements of the space $\mathrm{Hom}(V, V((z^{1/(n+1)})))$.

By taking into account this remark, from now on, we shall call a pair $(a(z), b(z))$, where $a(z) \in \mathrm{Hom}(V, V((z)))$, $b(z) \in \mathrm{Hom}(V, V((z^{1/(n+1)})))$, quasi compatible if there exists a nonzero polynomial $p(z_1, z_2)$ such that

$$p(z_1, z_2)a(z_1)b(z_2) \in \mathrm{Hom}(V, V((z_1, z_2^{1/(n+1)}))).$$

Remark 2.4 For an arbitrary polynomial $p(z_1, z_2) \neq 0$ there exist only finitely many $t \in \frac{1}{2}\mathbb{Z}$ such that $p(zq^t, z) = 0$. Hence, for any quasi compatible pair $(a(z), b(z))$ in $\mathrm{Hom}(V, V((z^{1/(n+1)})))$ there are only finitely many t such that $a(zq^t)_0 b(z) \neq 0$.

Set

$$V = K(1) \otimes \mathbb{C}\{P\}.$$

For $a(z) \in \mathrm{Hom}(V, V((z^{1/(n+1)})))$ we shall write

$$\mathrm{wt}\, a(z) = \alpha \in P$$

if $a(z)1 = b(z) \otimes e^{\alpha}$ for some $b(z) \in K(1)((z^{1/(n+1)}))$. For example,

$$\mathrm{wt}\, x_i^{\pm}(z) = \pm\alpha_i, \quad \mathrm{wt}\, \psi_i = \mathrm{wt}\, \phi_i = 0, \quad \mathrm{wt}\, \mathcal{Y}_i(z) = \lambda_i.$$

Definition 2.5 Let $(a(z), b(z))$ be a quasi compatible pair. Define

$$a(z)_{\bullet} b(z) = \sum_{t\in\frac{1}{2}\mathbb{Z}} \left(\frac{1}{zq^t}\right)^{(\mathrm{wt}\, a(z), \mathrm{wt}\, b(z))+1} q^t a(zq^t)_0 b(z). \tag{2.2}$$

Denote by $\langle \mathcal{Y}_i(z) \rangle$ a vector space over $\mathbb{C}(q^{1/2})$ spanned by all the vectors obtained by the above defined action of the operators $x_j^{\pm}(z)$, $\psi_j(z)$, $\phi_j(z)$ on $\mathcal{Y}_i(z)$:

$$\langle \mathcal{Y}_i(z) \rangle = \operatorname*{span}_{\mathbb{C}(q^{1/2})} \big\{ a_1(z)_{\bullet} \ldots_{\bullet} a_k(z)_{\bullet} \mathcal{Y}_i(z) \,:\, k \geq 0,\, a_s = x_j^{\pm}(z), \psi_j(z), \phi_j(z),$$
$$s = 1, 2, \ldots, k,\, j = 1, 2, \ldots, n \big\}.$$

2.2. **Space $\langle \mathcal{Y}_i(z)\rangle$.** First, we shall consider an action of the operators $x_j^-(z)$ on $\mathcal{Y}_i(z)$. Denote by $L(\lambda_i)$ the irreducible highest weight $U_q(\mathfrak{sl}_{n+1})$-module with the highest weight λ_i and with the highest weight vector v_{λ_i}.

Remark 2.6 On every $L(\lambda_i)$ we have $f_{j_k}\dots f_{j_2}f_{j_1}v_{\lambda_i}\neq 0$ if and only if $f_{j_{k-1}}\dots f_{j_2}f_{j_1}v_{\lambda_i}\neq 0$ and

$$(\lambda_i-\alpha_{j_1}-\dots-\alpha_{j_{k-1}},-\alpha_{j_k})=-1. \tag{2.3}$$

Similarly, $e_{j_k}f_{j_{k-1}}\dots f_{j_2}f_{j_1}v_{\lambda_i}\neq 0$ if and only if $f_{j_{k-1}}\dots f_{j_2}f_{j_1}v_{\lambda_i}\neq 0$ and

$$(\lambda_i-\alpha_{j_1}-\dots-\alpha_{j_{k-1}},\alpha_{j_k})=-1. \tag{2.4}$$

Lemma 2.7 *Let $k\in\mathbb{N}$. Then*

$$x_{j_k}^-(z)_\bullet\dots{}_\bullet x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet\mathcal{Y}_i(z)\neq 0 \quad \textit{if and only if}\quad f_{j_k}\dots f_{j_2}f_{j_1}v_{\lambda_i}\neq 0.$$

Proof. We shall prove by induction the following statement:

$$f_{j_k}\dots f_{j_2}f_{j_1}v_{\lambda_i}\neq 0$$

if and only if there exists a unique sequence $(t_k,\dots,t_1)$ in $\frac{1}{2}\mathbb{Z}$ such that

$$x_{j_l}^-(q^{t_l}z)_0x_{j_{l-1}}^-(z)_\bullet\dots{}_\bullet x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet\mathcal{Y}_i(z)\neq 0 \quad\text{for all } l=1,2,\dots,k.$$

Let $k=1$. Then $f_{j_1}v_{\lambda_i}\neq 0$ if and only if $j_1=i$. For $j_1=i$ relation (1.6) implies that $x_i^-(qz)_0\mathcal{Y}_i(z)\neq 0$ and $x_i^-(q^tz)_0\mathcal{Y}_i(z)=0$ for $t\neq 1$, so $x_i^-(z)_\bullet\mathcal{Y}_i(z)\neq 0$. For $j_1\neq i$ relation (1.7) implies that $x_{j_1}^-(q^tz)_0\mathcal{Y}_i(z)=0$ for all t. Therefore, the statement holds for $k=1$ and we have a unique sequence $(t_1)=(1)$.

Let $k=2$. Then $f_{j_2}f_iv_{\lambda_i}\neq 0$ if and only if $|j_2-i|=1$. For $j_2=i$ relations (1.3) and (1.6) imply that $x_i^-(q^tz)_0x_i^-(z)_\bullet\mathcal{Y}_i(z)=0$ for all t. For $|j_2-i|>1$ relations (1.5) and (1.7) imply that $x_{j_2}^-(q^tz)_0x_i^-(z)_\bullet\mathcal{Y}_i(z)=0$ for all t. Hence $x_{j_2}^-(z)_\bullet x_i^-(z)_\bullet\mathcal{Y}_i(z)=0$ for $|j_2-i|\neq 1$. For $|j_2-i|=1$ relations (1.4) and (1.7) imply that $x_{j_2}^-(q^2z)_0x_i^-(z)_\bullet\mathcal{Y}_i(z)\neq 0$ and $x_{j_2}^-(q^tz)_0x_i^-(z)_\bullet\mathcal{Y}_i(z)=0$ for $t\neq 2$. Therefore, the statement holds for $k=2$ and we have a unique sequence $(t_2,t_1)=(2,1)$.

Suppose that our statement holds for some $k\geq 2$. Assume that $f_{j_{k+1}}\dots f_{j_1}v_{\lambda_i}\neq 0$. Let

$$r=\max\left(\{l\leq k: j_l=j_{k+1}\}\cup\{0\}\right).$$

If $r=0$, then $j_{k+1}\notin\{j_k,\dots,j_1,i\}$. By condition (2.3) we see that there exists exactly one $l\in\{1,2,\dots,k\}$ such that $|j_{k+1}-j_l|=1$. Relations (1.4), (1.5) and (1.7) imply that

$$\begin{aligned}&x_{j_{k+1}}^-(q^{t_l+1}z)_0x_{j_k}^-(z)_\bullet\dots x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet\mathcal{Y}_i(z)\neq 0,\\ &x_{j_{k+1}}^-(q^tz)_0x_{j_k}^-(z)_\bullet\dots x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet\mathcal{Y}_i(z)=0\quad\text{for } t\neq t_l+1.\end{aligned}$$

Therefore, $x_{j_{k+1}}^-(z)_\bullet x_{j_k}^-(z)_\bullet\dots x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet\mathcal{Y}_i(z)\neq 0$ and we have a unique sequence $(t_l+1,t_k,t_{k-1},\dots,t_1)$.

If $r>0$, then condition (2.3) implies (together with the induction assumption) that there exist exactly two integers l_1 and l_2, $k\geq l_2>l_1>r$, such that $t_{l_1}=t_{l_2}=t_r+1$, $|j_{k+1}-j_{l_1}|=|j_{k+1}-j_{l_2}|=1$ and $j_{l_1}\neq j_{l_2}$. By considering relations (1.3)–(1.7) we see that

$$\begin{aligned}&x_{j_{k+1}}^-(q^{t_{l_1}+1}z)_0x_{j_k}^-(z)_\bullet\dots x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet\mathcal{Y}_i(z)\neq 0,\\ &x_{j_{k+1}}^-(q^tz)_0x_{j_k}^-(z)_\bullet\dots x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet\mathcal{Y}_i(z)=0\quad\text{for } t\neq t_{l_1}+1.\end{aligned}$$

Therefore, $x_{j_{k+1}}^-(z)_\bullet x_{j_k}^-(z)_\bullet\dots x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet\mathcal{Y}_i(z)\neq 0$ and we have a unique sequence $(t_{l_1}+1,t_k,t_{k-1},\dots,t_1)$.

Now assume that there exists a unique sequence $(t_{k+1},\dots,t_1)$ in $\frac{1}{2}\mathbb{Z}$ such that

$$x_{j_l}^-(q^{t_l}z)_0x_{j_{l-1}}^-(z)_\bullet\dots x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet\mathcal{Y}_i(z)\neq 0\quad\text{for all } l=1,2,\dots,k+1.$$

By induction assumption we have $f_{j_k}\dots f_{j_2}f_{j_1}v_{\lambda_i}\neq 0$.

If $r=0$, then by relations (1.3)–(1.7) we conclude that if $i<j_{k+1}$, then $j_{k+1}+1\notin\{j_k,\dots,j_1,i\}$ and that if $i>j_{k+1}$, then $j_{k+1}-1\notin\{j_k,\dots,j_1,i\}$. Hence, exactly one of two indices $j_{k+1}\pm 1$ is an element of the set $\{j_k,\dots,j_1,i\}$. Furthermore, there exists exactly one index $l=1,2,\dots,k$ such that $j_{k+1}-1=j_l$ or $j_{k+1}+1=j_l$, which implies that condition (2.3) holds, i.e.

$$(\lambda_i-\alpha_{j_1}-\dots-\alpha_{j_k},-\alpha_{j_{k+1}})=-1,$$

so $f_{j_{k+1}}\dots f_{j_2}f_{j_1}v_{\lambda_i}\neq 0$.

If $r>0$, then by induction assumption we have

$$(\lambda_i-\alpha_{j_1}-\dots-\alpha_{j_{r-1}},-\alpha_{j_{k+1}})=-1.$$

Since

$$x^-_{j_{k+1}}(q^tz)_0x^-_{j_r}(z)_\bullet\dots x^-_{j_2}(z)_\bullet x^-_{j_1}(z)_\bullet\mathcal{Y}_i(z)=0\quad\text{for all } t,$$

by considering relations (1.3)–(1.7) we conclude that there exist exactly two integers l_1 and l_2, $k\geq l_2>l_1>r$, such that $t_{l_1}=t_{l_2}=t_r+1$, $|j_{k+1}-j_{l_1}|=|j_{k+1}-j_{l_2}|=1$ and $j_{l_1}\neq j_{l_2}$. Hence, we have $(\lambda_i-\alpha_{j_1}-\dots-\alpha_{j_k},-\alpha_{j_{k+1}})=-1$ so the statement of the Lemma follows. □

Notice that for $x^-_{j_k}(z)_\bullet\dots{}_\bullet x^-_{j_1}(z)_\bullet\mathcal{Y}_i(z)\neq 0$ we have

$$\operatorname{wt} x^-_{j_k}(z)_\bullet\dots{}_\bullet x^-_{j_1}(z)_\bullet\mathcal{Y}_i(z)=\lambda_1-\alpha_{j_1}-\dots-\alpha_{j_k}=\operatorname{wt} f_{j_k}\dots f_{j_2}f_{j_1}v_{\lambda_i}.$$

Denote by $\mathcal{B}_i^-$ the set of all

$$x^-_{j_k}(z)_\bullet\dots{}_\bullet x^-_{j_1}(z)_\bullet\mathcal{Y}_i(z)\neq 0,\quad\text{where } j_s=1,2,\dots,n,\ s=1,2,\dots,k,\ k\in\mathbb{Z}_{\geq 0},$$

and denote by $\langle\mathcal{Y}_i(z)\rangle^-$ a subspace of $\langle\mathcal{Y}_i(z)\rangle$ spanned by $\mathcal{B}_i^-$. By Lemma 2.7 we have

$$\operatorname{wt}\mathcal{B}_i^-=\operatorname{wt}L(\lambda_i).$$

Remark 2.8 Considering (1.5) we see that if

$$f_{j_k}\dots f_{j_{p+1}}f_{j_p}\dots f_{j_1}v_{\lambda_i}=f_{\sigma(j_k)}\dots f_{j_{\sigma(j_{p+1})}}f_{j_p}\dots f_{j_1}v_{\lambda_i}$$

for some permutation σ, then

$$\begin{aligned}&x^-_{j_k}(z)_\bullet\dots{}_\bullet x^-_{j_{p+1}}(z)_\bullet x^-_{j_p}(z)_\bullet\dots{}_\bullet x^-_{j_1}(z)_\bullet\mathcal{Y}_i(z)\\&\quad=x^-_{\sigma(j_k)}(z)_\bullet\dots{}_\bullet x^-_{\sigma(j_{p+1})}(z)_\bullet x^-_{j_p}(z)_\bullet\dots{}_\bullet x^-_{j_1}(z)_\bullet\mathcal{Y}_i(z).\end{aligned}$$

Therefore, we have

$$\operatorname{card}\mathcal{B}_i^-=\dim L(\lambda_i).$$

Lemma 2.9 *The set $\mathcal{B}_i^-$ forms a basis of $\langle\mathcal{Y}_i(z)\rangle^-$.*

Proof. Suppose that

$$\sum_{s=1}^k\nu_sa_s(z)=0$$

for some nonzero scalars ν_s and $a_s(z)\in\mathcal{B}_i^-$. Furthermore, assume that $k\geq 2$ is the smallest positive integer for which such a nontrivial linear combination exists. We can choose p such that $\operatorname{wt}a_p(z)$ is maximal, i.e. $\operatorname{wt}a_p(z)\nless\operatorname{wt}a_r(z)$ for all $r\in\{1,2,\dots,k\}$. Next, we can choose $j_l,\dots,j_1$ such that $\operatorname{wt}x^-_{j_l}(z)_\bullet\dots{}_\bullet x^-_{j_1}(z)_\bullet a_p(z)$ is the lowest weight of $L(\lambda_i)$ and, therefore,

$$x^-_{j_l}(z)_\bullet\dots{}_\bullet x^-_{j_1}(z)_\bullet a_p(z)\neq 0.$$

Then the linear combination

$$x^-_{j_l}(z)_\bullet \dots {}_\bullet x^-_{j_1}(z)_\bullet \sum_{s=1}^{k} \nu_s a_s(z) = \sum_{s=1}^{k} \nu_s x^-_{j_l}(z)_\bullet \dots {}_\bullet x^-_{j_1}(z)_\bullet a_s(z) = 0$$

consists of less then k nonzero summands. Contradiction. $\square$

Next, we have an analogue of Lemma 2.7 for the operators $x^+_j(z)$:

Lemma 2.10 *Let $k \in \mathbb{N}$. Then*

$$x^+_{j_{k+1}}(z)_\bullet x^-_{j_k}(z)_\bullet \dots {}_\bullet x^-_{j_2}(z)_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) \neq 0 \quad \textit{if and only if} \quad e_{j_{k+1}} f_{j_k} \dots f_{j_2} f_{j_1} v_{\lambda_i} \neq 0.$$

Proof. Let

$$r = \max\left(\{l : j_l = j_{k+1}\} \cup \{0\}\right).$$

If $r = 0$, then $e_{j_{k+1}} f_{j_k} \dots f_{j_1} v_{\lambda_i} = 0$ and, by relations (1.14)–(1.17),

$$x^+_{j_{k+1}}(z)_\bullet x^-_{j_k}(z)_\bullet \dots {}_\bullet x^-_{j_2}(z)_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) = 0.$$

If $r \neq 0$, then there exists a unique integer t_r such that

$$\begin{aligned} &x^-_{j_r}(q^{t_r} z)_0 x^-_{j_{r-1}}(z)_\bullet \dots {}_\bullet x^-_{j_2}(z)_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) \neq 0, \\ &x^-_{j_r}(q^{t} z)_0 x^-_{j_{r-1}}(z)_\bullet \dots {}_\bullet x^-_{j_2}(z)_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) = 0, \quad \text{for } t \neq t_r \end{aligned}$$

(cf. the proof of Lemma 2.7). If there exists an integer s such that $r < s \leq k$ and $|j_s - j_r| = 1$, then by (2.3) and (2.4) we get $e_{j_{k+1}} f_{j_k} \dots f_{j_1} v_{\lambda_i} = 0$ and by relations (1.13)–(1.17)

$$x^+_{j_{k+1}}(z)_\bullet x^-_{j_k}(z)_\bullet \dots {}_\bullet x^-_{j_2}(z)_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) = 0.$$

If $|j_s - j_r| > 1$ for all integers s such that $r < s \leq k$, then by (2.3) and (2.4) we get $e_{j_{k+1}} f_{j_k} \dots f_{j_1} v_{\lambda_i} \neq 0$ and relations (1.13)–(1.17) imply that

$$\begin{aligned} &x^+_{j_{k+1}}(q^{t_r+1} z)_0 x^-_{j_k}(z)_\bullet \dots {}_\bullet x^-_{j_2}(z)_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) \neq 0, \\ &x^+_{j_{k+1}}(q^{t} z)_0 x^-_{j_k}(z)_\bullet \dots {}_\bullet x^-_{j_2}(z)_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) = 0, \quad \text{for } t \neq t_r + 1, \end{aligned}$$

so the statement of the lemma follows. $\square$

By considering relations (1.8)–(1.12) and applying the same technique as in the proof of Lemmas 2.7 and 2.10 one can prove

Lemma 2.11 *Let $k \in \mathbb{N}$. Then*

$$\psi_{j_{k+1}}(z)_\bullet x^-_{j_k}(z)_\bullet \dots {}_\bullet x^-_{j_2}(z)_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) \neq 0$$

if and only if

$$e_{j_{k+1}} f_{j_k} \dots f_{j_2} f_{j_1} v_{\lambda_i} \neq 0 \quad \textit{or} \quad f_{j_{k+1}} f_{j_k} \dots f_{j_2} f_{j_1} v_{\lambda_i} \neq 0.$$

As before, if $\psi_{j_{k+1}}(z)_\bullet x^-_{j_k}(z)_\bullet \dots {}_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) \neq 0$, then there exists a unique $t \in \frac{1}{2}\mathbb{Z}$ such that $\psi_{j_{k+1}}(zq^t)_0 x^-_{j_k}(z)_\bullet \dots {}_\bullet x^-_{j_1}(z)_\bullet \mathcal{Y}_i(z) \neq 0$.

Since $\phi_j(z)$ contains only nonnegative powers of the variable z, we have

Lemma 2.12 *For any $a(z) \in \langle \mathcal{Y}_i(z) \rangle$ and $j = 1, 2, \dots, n$*

$$\phi_j(z)_\bullet a(z) = 0.$$

In order to clarify the application of Definition 2.5 on the Frenel-Jing operators, we list in the following corollary necessary conditions for the summands on the right side of (2.2) to be nonzero. Its statement is a consequence of the proofs of Lemmas 2.7, 2.10 and 2.11.

Corollary 2.13 *Let $j = 1, 2, \ldots, n$ and*

$$a(zq^t)_0 x^-_{j_k}(zq^{t_k})_0 \ldots x^-_{j_1}(zq^{t_1})_0 \mathcal{Y}_i(z) \neq 0.$$

Set $t_0 = 0$ and

$$r_0 = \max\left(\{l : |j_l - j| = 1\} \cup \{0\}\right).$$

Then

$$t = \begin{cases} t_{r_0} + 1 & \text{for } a(z) = x^-_j(z); \\ t_{r_0} + \frac{3}{2} & \text{for } a(z) = \psi_j(z); \\ t_{r_0} + 2 & \text{for } a(z) = x^+_j(z). \end{cases}$$

The next lemma gives us an analogue of Drinfeld relation (D9) on the space $\langle \mathcal{Y}_i(z) \rangle$.

Lemma 2.14 *For any $a(z) \in \langle \mathcal{Y}_i(z) \rangle$ we have*

$$x^+_{j_1}(z)_\bullet x^-_{j_2}(z)_\bullet a(z) - x^-_{j_2}(z)_\bullet x^+_{j_1}(z)_\bullet a(z) = \frac{\delta_{j_1 j_2}}{q - q^{-1}} \left(\psi_{j_1}(z) - \phi_{j_1}(z)\right)_\bullet a(z). \tag{2.5}$$

Proof. By using Lemmas 2.7 and 2.10 together with (2.3) and (2.4), we easily see that $[x^+_{j_1}(z), x^-_{j_2}(z)] = 0$ on $\langle \mathcal{Y}_i(z) \rangle$ for $j_1 \neq j_2$.

Assume $j_1 = j_2$. Set $j = j_1 = j_2$ and let

$$a(z) = x^-_{p_k}(z)_\bullet \ldots {}_\bullet x^-_{p_1}(z)_\bullet \mathcal{Y}_i(z)$$

be an element of $\langle \mathcal{Y}_i(z) \rangle$ such that $x^-_j(zq^t)_0 a(z) \neq 0$ for some (unique) t. By using relations given by Proposition 1.5 one can construct polynomials $r_s(z_1, z_2)$, $s = 1, 2, 3$, such that

$$q^t(z_1 - z_2) r_1(z_1, z_2) x^-_j(z_1 q^t) a(z_2) = r_1(z_1, z_2) : x^-_j(z_1 q^t) a(z_2) : \tag{2.6}$$

$$q^{t+1}(z_1 - z_2) r_2(z_1, z_2) x^+_j(z_1 q^{t+1}) x^-_j(z_2)_\bullet a(z_2) = r_2(z_1, z_2) : x^+_j(z_1 q^{t+1}) x^-_j(z_2)_\bullet a(z_2) : \tag{2.7}$$

$$(z_1 - z_2) r_3(z_1, z_2) \psi_j(z_1 q^{t+1/2}) a(z_2) = (q z_1 - q^{-1} z_2) r_3(z_1, z_2) a(z_2) \psi_j(z_1 q^{t+1/2}). \tag{2.8}$$

Next, we have

$$: x^+_j(q^{t+1} z) x^-_j(q^t z) := \psi_j(z q^{t+1/2}), \tag{2.9}$$

so relations (2.6)–(2.8), together with (2.9), imply that equation (2.5) holds.

For $a(z) = x^-_{p_k}(z)_\bullet \ldots {}_\bullet x^-_{p_1}(z)_\bullet \mathcal{Y}_i(z)$ such that $x^+_j(z)_\bullet a(z) \neq 0$ we can proceed similarly and for $a(z) = x^-_{p_k}(z)_\bullet \ldots {}_\bullet x^-_{p_1}(z)_\bullet \mathcal{Y}_i(z)$ such that $x^\pm_j(z)_\bullet a(z) = 0$ the statement follows from Lemma 2.11. □

Since $\mathrm{wt}\, \psi_j(z) = 0$ we have

$$\mathrm{wt}\, \langle \mathcal{Y}_i(z) \rangle = \mathrm{wt}\, \mathcal{B}^-_i = \mathrm{wt}\, L(\lambda_i).$$

Every homogeneous vector $a(z) \in \langle \mathcal{Y}_i(z) \rangle$ is a linear combination of the vectors having the form

$$b(z) \psi_{j_1}(zq^{t_{j_1}}) \ldots \psi_{j_k}(zq^{t_{j_k}})$$

for

$$b(z) \in \mathcal{B}^-_i, \ \ k \geq 0, \ \ 1 \leq j_1 \leq \ldots \leq j_k \leq n, \ \ t_{j_s} \in \tfrac{1}{2}\mathbb{Z}, \ \ \mathrm{wt}\, a(z) = \mathrm{wt}\, b(z).$$

Denote by $\mathcal{B}^-_{i,\psi}$ the set of all vectors

$$b(z) = b^-(z) \psi_{j_1}(zq^{t_1}) \ldots \psi_{j_k}(zq^{t_k}), \tag{2.10}$$

where

$$\begin{aligned} &b^-(z) \in \mathcal{B}^-_i, \ \ k \geq 0, \ \ 1 \leq j_1 \leq \ldots \leq j_k \leq n, \ \ t_s \in \tfrac{1}{2}\mathbb{Z}, \\ &t_s \leq t_r \text{ if } s \leq r \text{ and } j_s = j_r. \end{aligned}$$

Set

$$o(b(z)) = (t_1, \ldots, t_k), \qquad t(b(z)) = (j_1, \ldots, j_k). \tag{2.11}$$

Naturally, the space $\langle \mathcal{Y}_i(z) \rangle_\psi$ spanned by the set $\mathcal{B}^-_{i,\psi}$ (over $\mathbb{C}(q^{1/2})$) is bigger than $\langle \mathcal{Y}_i(z) \rangle$, i.e. $\langle \mathcal{Y}_i(z) \rangle \subsetneq \langle \mathcal{Y}_i(z) \rangle_\psi$. For example,

$$\mathcal{Y}_1(z)\psi_2(zq^t) \notin \langle \mathcal{Y}_1(z) \rangle \quad \text{for all } t \in \tfrac{1}{2}\mathbb{Z}$$

because

$$\psi_2(z)_\bullet \mathcal{Y}_1(z) = 0. \tag{2.12}$$

Lemma 2.15 *The set $\mathcal{B}^-_{i,\psi}$ is linearly independent.*

Proof. We shall prove a "stronger" statement from which the statement of the Lemma clearly follows: The set of all vectors

$$b(z)\psi_{j_1}(zq^{t_1}) \ldots \psi_{j_k}(zq^{t_k}),$$

where

$$b(z) \in \bigcup_{i=1}^n \mathcal{B}^-_i, \;\; k \geq 0, \;\; 1 \leq j_1 \leq \ldots \leq j_k \leq n, \;\; t_s \in \tfrac{1}{2}\mathbb{Z},$$
$$t_s \leq t_r \quad \text{if} \quad s \leq r \text{ and } j_s = j_r.$$

is linearly independent.

Suppose that

$$\sum_{s=1}^k \nu_s a_s(z) = 0 \quad \text{for some } \nu_s \in \mathbb{C}(q^{1/2}) \setminus \{0\}\,,\, a_s(z) \in \bigcup_{i=1}^n \mathcal{B}^-_{i,\psi}. \tag{2.13}$$

Furthermore, assume that $k \geq 2$ is the smallest positive integer for which such a nontrivial linear combination exists. Without loss of generality we can assume that the weights of all $a_s(z)$ are equal. Indeed, if $\operatorname{wt} a_r(z) \neq \operatorname{wt} a_s(z)$ for some $r, s = 1, 2, \ldots, k$, we can proceed similarly as in the proof of Lemma 2.9. Next, we can assume that each $a_s(z)$ is of the form

$$a_s(z) = \mathcal{Y}_i(z)\psi_{j_{1,s}}(zq^{t_{1,s}}) \cdots \psi_{j_{l_s,s}}(zq^{t_{l_s,s}})$$

for some $l_s \geq 0$, $1 \leq j_{1,s} \leq \ldots \leq j_{l_s,s} \leq n$, $t_{j,s} \in \frac{1}{2}\mathbb{Z}$, $j = 1, 2, \ldots, l_s$. Multiplying linear combination (2.13) by appropriate invertible operators we can "remove" $\mathcal{Y}_i(z)$ and get

$$L(z) := \sum_{s=1}^k \nu_s \psi_{j_{1,s}}(zq^{t_{1,s}}) \cdots \psi_{j_{l_s,s}}(zq^{t_{l_s,s}}) = 0. \tag{2.14}$$

Since

$$e^{\lambda_m}\psi_j(z) = q^{-\delta_{m,j}}\psi_j(z)e^{\lambda_m},$$

by multiplying (2.14) with e^{λ_m}, moving the (invertible) operator e^{λ_m} all the way to the right and then dropping the operator we get

$$\sum_{s=1}^k \nu_s q^{-\delta_{m,j_{1,s}} - \ldots - \delta_{m,j_{l_s,s}}} \psi_{j_{1,s}}(zq^{t_{1,s}}) \cdots \psi_{j_{l_s,s}}(zq^{t_{l_s,s}}) = 0. \tag{2.15}$$

From (2.15) we see that it is sufficient to consider only a linear combination as in (2.14) such that all the powers of q, $-\delta_{m,j_{1,s}} - \ldots - \delta_{m,j_{l_s,s}}$, $s = 1, 2, \ldots, k$, are equal. Since the operators $\psi_j(z)$ are invertible, we can assume that for each j and t there exists an index s such that the operator $\psi_j(zq^t)$ does not appear in the sth summand in (2.14). Recall

relations (1.11) and (1.12). We can choose some $j = 1, 2, \ldots, n$ and $t \in \frac{1}{2}\mathbb{Z}$ such that there exist indices r and s such that

$$\psi_{j_{1,r}}(zq^{t_{1,r}}) \cdots \psi_{j_{l_r,r}}(zq^{t_{l_r,r}})_{\bullet}\mathcal{Y}_j(zq^t) = 0 \quad \text{and} \quad \psi_{j_{1,s}}(zq^{t_{1,s}}) \cdots \psi_{j_{l_s,s}}(zq^{t_{l_s,s}})_{\bullet}\mathcal{Y}_j(zq^t) \neq 0.$$

Finally, by using the substitution $z_0 = q^{-t}z$ in $L(z_0)_{\bullet}\mathcal{Y}_j(z_0q^t) = 0$ we get a contradiction to the choice of k. □

By the discussion preceding Lemma 2.15 we see that the set

$$\mathcal{B}_i = \mathcal{B}_{i,\psi}^- \cap \langle \mathcal{Y}_i(z) \rangle$$

spans $\langle \mathcal{Y}_i(z) \rangle$, so as a consequence of Lemma 2.15 we have

Theorem 2.16 *The set $\mathcal{B}_i$ forms a basis for the space $\langle \mathcal{Y}_i(z) \rangle$.*

Example 1 Consider the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. The basis for the space $\langle \mathcal{Y}_1(z) \rangle$, given in Theorem 2.16, is the set

$$\mathcal{B}_1 = A_{1,\lambda_1} \cup A_{1,\lambda_1-\alpha_1},$$

where

$$\begin{aligned} A_{1,\lambda_1} &= \left\{ \mathcal{Y}_1(z)\psi_1(zq^{3/2})^l : l \in \mathbb{Z}_{\geq 0} \right\}; \\ A_{1,\lambda_1-\alpha_1} &= \left\{ x_1^-(z)_{\bullet}\mathcal{Y}_1(z)\psi_1(zq^{3/2})^l : l \in \mathbb{Z}_{\geq 0} \right\}. \end{aligned}$$

Example 2 Consider the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_3)$. The basis for the space $\langle \mathcal{Y}_1(z) \rangle$, given in Theorem 2.16, is the set

$$\mathcal{B}_1 = A_{2,\lambda_1} \cup A_{2,\lambda_1-\alpha_1} \cup A_{2,\lambda_1-\alpha_1-\alpha_2},$$

where

$$\begin{aligned} A_{2,\lambda_1} &= \left\{ \mathcal{Y}_1(z)\psi_1(zq^{3/2})^l\psi_2(zq^{5/2})^m : l, m \in \mathbb{Z}_{\geq 0}, \text{ if } m > 0 \text{ then } l > 0 \right\}; \\ A_{2,\lambda_1-\alpha_1} &= \left\{ x_1^-(z)_{\bullet}\mathcal{Y}_1(z)\psi_1(zq^{3/2})^l\psi_2(zq^{5/2})^m : l, m \in \mathbb{Z}_{\geq 0} \right\}; \\ A_{2,\lambda_1-\alpha_1-\alpha_2} &= \left\{ x_2^-(z)_{\bullet}x_1^-(z)_{\bullet}\mathcal{Y}_1(z)\psi_1(zq^{3/2})^l\psi_2(zq^{5/2})^m : l, m \in \mathbb{Z}_{\geq 0} \right\}. \end{aligned}$$

The constraint "if $m > 0$ then $l > 0$", in the definition of set A_{2,λ_1}, is a consequence of (2.12).

3. Algebra $U_q(\mathfrak{sl}_{n+1})_z$ and its representations

3.1. **Algebra $U_q(\mathfrak{sl}_{n+1})_z$.** For $j = 1, 2, \ldots, n$ define elements $L_j, M_j \in U_q(\mathfrak{h}) \subset U_q(\mathfrak{sl}_{n+1})$ by

$$\begin{aligned} L_j &= K_1^{n+1-j}K_2^{2(n+1-j)} \cdots K_{j-1}^{(j-1)(n+1-j)}K_j^{j(n+1-j)}K_{j+1}^{j(n-j)} \cdots K_n^j, \\ M_j &= \begin{cases} q^{-(n+1)^2} & \text{if } j = 1 \text{ or } j = n, \\ q^{-(n+1)^2}L_{j-1}L_{j+1} & \text{if } 1 < j < n. \end{cases} \end{aligned}$$

The elements L_j and M_j are invertible and they satisfy

$$L_jL_k = L_kL_j, \qquad M_jM_k = M_kM_j$$

for $j, k = 1, 2, \ldots, n$. Let $z_1, \ldots, z_n$ be commutative formal variables and

$$w_j = e^{(M_j - M_j^{-1})z_j} = \sum_{r \geq 0} \frac{(M_j - M_j^{-1})^r}{r!} z_j^r \in U_q(\mathfrak{sl}_{n+1})[[z_1, \ldots, z_n]], \quad j = 1, 2, \ldots, n.$$

The algebra $U_q(\mathfrak{h})[w_1, \ldots, w_n]$ is commutative but the elements w_j are not central in the algebra $U_q(\mathfrak{sl}_{n+1})[w_1, \ldots, w_n]$.

Define elements $\bar{e}_j, \bar{k}_j \in U_q(\mathfrak{sl}_{n+1})[w_1, \ldots, w_n]$ by

$$\bar{e}_j = e_j w_j, \quad \bar{k}_j = (K_j - K_j^{-1}) w_j \qquad \text{for } j = 1, 2, \ldots, n.$$

Denote by $U_q(\mathfrak{sl}_{n+1})_z$ a $\mathbb{C}(q^{1/2})$ subalgebra of $U_q(\mathfrak{sl}_{n+1})[w_1, \ldots, w_n]$ generated by the elements

$$\bar{e}_j,\ f_j,\ \bar{k}_j, \quad j = 1, 2, \ldots, n.$$

Remark 3.1 Since the classical limit $q \to 1$ of K_j is equal to 1 (cf. [16]), the classical limit of w_j is also 1, so the classical limit of $U_q(\mathfrak{sl}_{n+1})_z$ is the universal enveloping algebra $U(\mathfrak{sl}_{n+1})$. Moreover, classical limits of $\bar{e}_j$, f_j and $\bar{h}_j = \bar{k}_j/(q - q^{-1})$ are exactly Chevalley generators of $U(\mathfrak{sl}_{n+1})$.

Remark 3.2 In general, the algebra $U_q(\mathfrak{sl}_{n+1})_z$ can not be defined in terms of closed-form expressions among generators $\bar{e}_j, f_j, k_j$, $j = 1, 2, \ldots, n$. For example, we have

$$f_{j+1} M_j = q^{n+1} M_j f_{j+1}$$

for $2 \leq j \leq n-1$ and therefore

$$\begin{aligned} f_{j+1}\bar{e}_j = f_{j+1} e_j w_j = e_j f_{j+1} w_j &= e_j f_{j+1} \sum_{r\geq 0} \frac{(M_j - M_j^{-1})^r}{r!} z_j^r \\ &= e_j \left(\sum_{r\geq 0} \frac{(q^{n+1} M_j - q^{-n-1} M_j^{-1})^r}{r!} z_j^r \right) f_{j+1}. \end{aligned}$$

3.2. **Module $L(\lambda_i)_z$.** For any $U_q(\mathfrak{sl}_{n+1})$-module V the space $V[[z_1, \ldots, z_n]]$ can be in a natural way equipped by the structure of $U_q(\mathfrak{sl}_{n+1})[[z_1, \ldots, z_n]]$-module, as well as by the structure of $U_q(\mathfrak{sl}_{n+1})[w_1, \ldots, w_n]$-module. Naturally, every $U_q(\mathfrak{sl}_{n+1})[w_1, \ldots, w_n]$-module is also a $U_q(\mathfrak{sl}_{n+1})_z$-module.

Let v_{λ_i} be the highest weight vector of $U_q(\mathfrak{sl}_{n+1})$-module $L(\lambda_i)$ with the dominant integral highest weight λ_i. Denote by $L(\lambda_i)_z$ an $U_q(\mathfrak{sl}_{n+1})_z$-submodule of $L(\lambda_i)[[z_1, \ldots, z_n]]$ generated by v_{λ_i}, i.e.

$$L(\lambda_i)_z = U_q(\mathfrak{sl}_{n+1})_z v_{\lambda_i}.$$

Theorem 3.3 *(1) There exists a structure of $U_q(\mathfrak{sl}_{n+1})_z$-module on the space $\langle \mathcal{Y}_i(z) \rangle$ such that*

$$\bar{e}_j a(z) = x_j^+(z)_\bullet a(z), \tag{3.1}$$

$$f_j a(z) = x_j^-(z)_\bullet a(z), \tag{3.2}$$

$$\bar{k}_j a(z) = \psi_j(z)_\bullet a(z) \tag{3.3}$$

for all $j = 1, 2, \ldots, n$ and $a(z) \in \langle \mathcal{Y}_i(z) \rangle$.
(2) $U_q(\mathfrak{sl}_{n+1})_z$-modules $L(\lambda_i)_z$ and $\langle \mathcal{Y}_i(z) \rangle$ are isomorphic.

Proof. First, notice that

$$\psi_j(z)_\bullet x_{j_k}^-(z)_\bullet \ldots_\bullet x_{j_2}^-(z)_\bullet x_{j_1}^-(z)_\bullet \mathcal{Y}_i(z) \neq 0 \quad \text{if and only if} \quad \bar{k}_j f_{j_k} \ldots f_{j_2} f_{j_1} v_{\lambda_i} \neq 0.$$

For any homogeneous vector $v \in L(\lambda_i)$ we have

$$w_j v = v e(u, z_j) \quad \text{for some } u \in \mathbb{Z},$$

where

$$e(u, z_j) = \exp((q^u - q^{-u}) z_j) = \sum_{r\geq 0} \frac{(q^u - q^{-u})^r}{r!} z_j^r \in \mathbb{C}(q^{1/2})[[z_1, \ldots, z_n]].$$

Let $\mathcal{C}$ be a set of all nonzero vectors

$$c = f_{l_1} \cdots f_{l_r} v_{\lambda_i} e(u_1, z_{j_1}) \cdots e(u_s, z_{j_s}),$$

where

$$l_1, \ldots, l_r \in \{1, 2, \ldots, n\}, \ \ 1 \leq j_1 \leq \ldots \leq j_s \leq n, \ \ u_1, \ldots, u_s \in \mathbb{Z}, \ \ r, s \geq 0,$$
$$u_l \geq u_m \quad \text{if} \quad l \leq m \text{ and } j_l = j_m.$$

Recall (2.11) and set

$$o(c) = (-u_1, \ldots, -u_s), \qquad t(c) = (j_1, \ldots, j_s).$$

The set $\mathcal{C}_i := \mathcal{C} \cap L(\lambda_i)_z$ forms a basis of the space $L(\lambda_i)_z$.

There exists a unique isomorphism of vector spaces $\Omega \colon L(\lambda_i)_z \to \langle \mathcal{Y}_i(z) \rangle$ satisfying the following two conditions:

(1) For all $f_{l_1} \cdots f_{l_r} v_{\lambda_i} e(u_1, z_{j_1}) \cdots e(u_s, z_{j_s}) \in \mathcal{C}_i$ there exist $t_1, \ldots, t_s \in \frac{1}{2}\mathbb{Z}$ such that

$$\begin{aligned} &\Omega(f_{l_1} \cdots f_{l_r} v_{\lambda_i} e(u_1, z_{j_1}) \cdots e(u_s, z_{j_s})) \\ &\qquad = x^-_{l_1}(z)_\bullet \ldots {}_\bullet x^-_{l_r}(z)_\bullet \mathcal{Y}_i(z) \psi_{j_1}(zq^{t_1}) \cdots \psi_{j_s}(zq^{t_s}) \in \mathcal{B}_i; \end{aligned}$$

(2) For all $c_1, c_2 \in \mathcal{C}_i$ we have

$$\text{if } \operatorname{wt}(c_1) = \operatorname{wt}(c_2), \ t(c_1) = t(c_2) \text{ and } o(c_1) \leq o(c_2) \quad \text{then} \quad o(\Omega(c_1)) \leq o(\Omega(c_2)),$$

where "$\leq$" is lexicographic order.

Notice that

$$\operatorname{wt}(\Omega(c)) = \operatorname{wt}(c)$$

for all $c \in \mathcal{C}_i$ and

$$u_l - u_m = -(n+1)(t_l - t_m) \quad \text{when} \quad j_l = j_m.$$

The actions of the operators $\bar{e}_j$, f_j, $\bar{k}_j$ on an arbitrary basis vector $c \in \mathcal{C}_i$ correspond to the actions of operators $x^+_j(z)$, $x^-_j(z)$, $\psi_j(z)$ on $b(z) = \Omega(c)$ respectively. For example, if $f_j c \neq 0$ for some $c \in \mathcal{C}_i$, then we have commutative diagrams as in Figure 1. The left diagram is a consequence of (Q3), while the right diagram is a consequence of Lemma 2.14.

$$\begin{tikzcd} ce(u,z_j) \arrow[r, "f_j"] & f_j ce(u,z_j) \\ c \arrow[u, "\frac{\bar{k}_j}{q-q^{-1}}"] \arrow[r, "f_j"] & f_j c \arrow[u, "-\frac{\bar{k}_j}{q-q^{-1}}"'] \arrow[ul, "\bar{e}_j"'] \end{tikzcd} \qquad \begin{tikzcd} b(z)\psi_j(zq^t) \arrow[r, "x^-_j(z)"] & x^-_j(z)_\bullet b(z)\psi_j(zq^t) \\ b(z) \arrow[u, "\frac{\psi_j(z)}{q-q^{-1}}"] \arrow[r, "x^-_j(z)"] & x^-_j(z)_\bullet b(z) \arrow[u, "-\frac{\psi_j(z)}{q-q^{-1}}"'] \arrow[ul, "x^+_j(z)"'] \end{tikzcd}$$

FIGURE 1. Commutative diagrams in $L(\lambda_i)_z$ and $\langle \mathcal{Y}_i(z) \rangle$

Finally, we conclude that formulas (3.1)–(3.3) define an $U_q(\mathfrak{sl}_{n+1})_z$-module structure on the space $\langle \mathcal{Y}_i(z) \rangle$, so the mapping Ω becomes an $U_q(\mathfrak{sl}_{n+1})_z$-module isomorphism. $\square$

Remark 3.4 For $i = 0$ we have $\mathcal{Y}_i(z) = \mathcal{Y}_0(z) = 1$. The action of the Frenkel-Jing operators on $\mathcal{Y}_0(z)$ is trivial, i.e.

$$a(z)_{\bullet}\mathcal{Y}_0(z) = 0 \quad \text{for } a(z) = x_j^{\pm}(z), \psi_j(z), \phi_j(z),\ j = 1, 2, \ldots, n,$$

so the space $\langle \mathcal{Y}_0(z) \rangle$ is one-dimensional.

In the end, we would like to provide an example of $U_q(\mathfrak{sl}_{n+1})_z$-modules. The $U_q(\mathfrak{sl}_{n+1})_z$-module $L(\lambda_i)_z$, when considered as a vector space over $\mathbb{C}(q^{1/2})$, has a weight decomposition

$$L(\lambda_i)_z = \bigoplus_{\mu \in \operatorname{wt} L(\lambda_i)} (L(\lambda_i)_z)_\mu\,, \quad \text{where} \quad (L(\lambda_i)_z)_\mu = \{v \in L(\lambda_i)_z : \operatorname{wt} v = \mu\}\,.$$

Its weight subspaces $(L(\lambda_i)_z)_\mu$, $\mu \in \operatorname{wt} L(\lambda_i)$, are infinite-dimensional.

Example 3 Consider the diagrams of $U_q(\mathfrak{sl}_2)_z$-module $L(\lambda_1)_z$ and $U_q(\mathfrak{sl}_3)_z$-module $L(\lambda_1)_z$ given in Figures 2 and 3 respectively. Each node represents one weight vector from the corresponding module. This vector is also an element of the basis $\mathcal{C}_1$, that was constructed in the proof of Theorem 3.3. Weight of each node is written as its label and all the nodes (i.e. corresponding vectors) are linearly independent. The yellow node represents the highest weight vector $v_{\lambda_1} \in L(\lambda_1) \subset L(\lambda_1)_z$. The arrows represent (nonzero) actions of the generators $\bar{e}_j, f_j, \bar{h}_j = \bar{k}_j/(q - q^{-1})$, where $j = 1$ (Figure 2) or $j = 1, 2$ (Figure 3), on the corresponding nodes (vectors). All the diagrams in the both figures are commutative.

The images of the bases $\mathcal{C}_1$, given in Figures 2 and 3, under the $U_q(\mathfrak{sl}_{n+1})_z$-module isomorphism Ω, constructed in the proof of Theorem 3.3, where $n = 1, 2$, are equal to the bases $\mathcal{B}_1$, given in Examples 1 and 2.

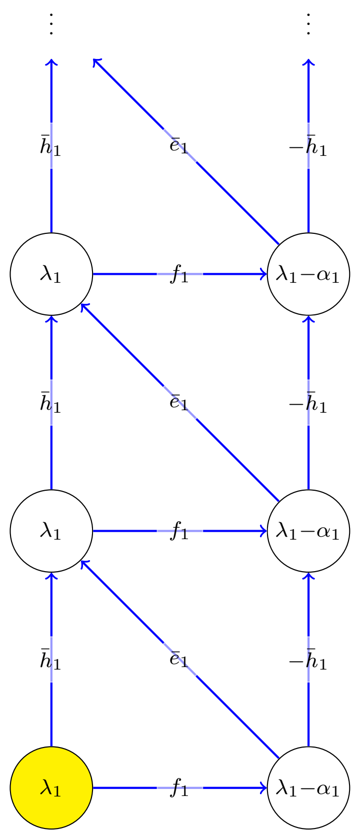

FIGURE 2. $U_q(\mathfrak{sl}_2)_z$-module $L(\lambda_1)_z$

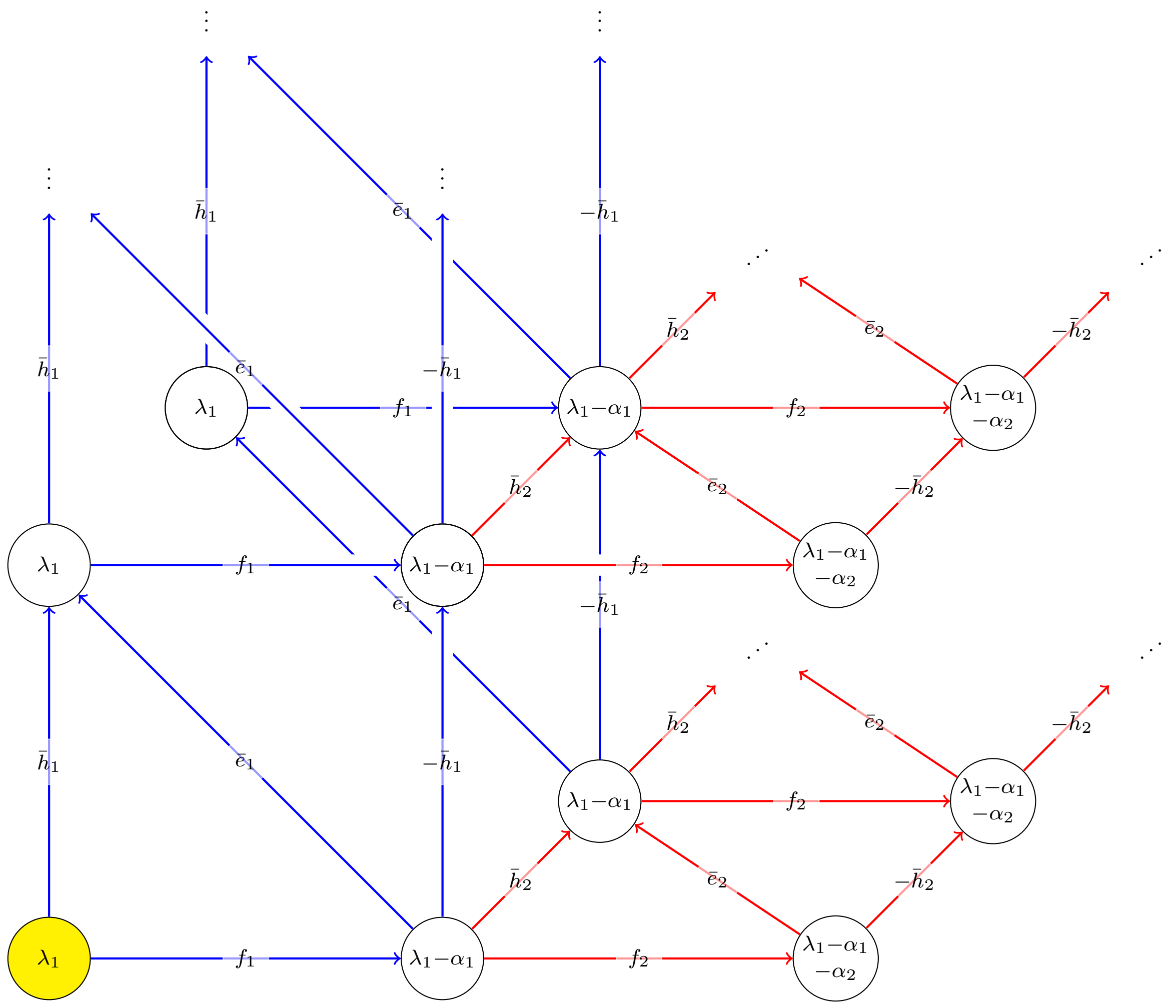

FIGURE 3. $U_q(\mathfrak{sl}_3)_z$-module $L(\lambda_1)_z$

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